One page proof of the Riemann hypothesis

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Abstract. We give a short Wiener measure proof of the Riemann hypothesis based on a surprising, unexpected and deep relation between the Riemann zeta $\zeta(s)$ and the trivial zeta $\zeta_t(s) := Im(s)(2Re(s) - 1)$.

1 Functional analysis and probability theory of the Riemann hypothesis.

Let $C^+ = C^+(\mathbb{R})$ be the real vector space of all real valued, **symmetric** and **continuous** functions defined on the **real numbers field** $\mathbb{R}$. We also consider the following **normed spaces**, **Banach spaces** and **Frechet spaces** .

By S_2 we denote the real vector subspace of functions $c \in C^+$ with the finite **second-max-moment**, i.e.

$$|| c ||_2 := maximum_{x\in\mathbb{R}} \mid x^2 c(x) \mid < +\infty,$$

whereas by S_0 we denote the real vector subspace of C^+ of all **bounded continuous functions** with the sup-norm $|| \cdot ||_0$. Obviously, $B_2 := (S_2, || \cdot ||_2)$ is a normed space and $B_0 := (S_0, || \cdot ||_0)$ is a Banach space.

Let us denote by I the unit interval $[0, 1]$ and by $I^c = (1, +\infty)$ its complement. Moreover, $L^1(I^c, dx)$ is the Lebesgue space of all real absolute integrable functions $f : I^c \longrightarrow \mathbb{C}$ with the **finite first integral moment**

$$|| f ||_1 := \int_{I^c} \mid f(x) \mid dx < +\infty,$$

and dx is the **Lebesgue measure**. In particular, since

$$\int_{I^c} \mid f(x) \mid dx \leq || f ||_2 \int_1^\infty \frac{dx}{x^2},$$

we have $S_2 \mid I^c \subset L^1(I^c, dx)$.

We consider the **canonical Fourier (cosine) transform** $\mathcal{F} : S_2 \longrightarrow S_0 \cap C^+$ defined by the well-known formula :

$$(\mathcal{F}f)(x) \; := \; \int_{\mathbb{R}} e^{2\pi ixy} f(y)dy = 2\int_0^{+\infty} cos(2\pi xy)f(y)dy =: \hat{f}(x), \tag{1.1}$$

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where $x \in \mathbb{R}_+ := [0, +\infty)$ and $f \in S_2$.

The formula

$$\theta(f)(x) := \sum_{n=1}^{\infty} f(nx) = \int_{\mathbb{N}^*} f(nx) dc_{\mathbb{Z}}(n); x > 0, f \in S_2, \tag{1.2}$$

defines the canonical **Jacobi theta transform** $\theta : S_2 \longrightarrow C^+(\mathbb{R} - \{0\})$, where $\mathbb{N}^*, \mathbb{N}$ and $\mathbb{Z}$ are the **multiplicative semigroup of positive integers**, **additive semigroup of positive integers** and the **ring of integers**, respectively and $c_{\mathbb{Z}}$ marks the **calculating (Haar) measure** of $(\mathbb{Z}, +)$ normalized as : $c_{\mathbb{Z}}(\{0\}) = 1$.

Finally, by $M : S_2 \longrightarrow C(0, 3] \otimes_{\mathbb{R}} \mathbb{C}$ we denote the **Mellin transform**, i.e.

$$M(f)(s) \ := \ \int_0^{\infty} x^s f(x) \frac{dx}{x} \ Re(s) > 0, f \in S. \tag{1.3}$$

All in the sequel $\mathbb{C}$ marks the **complex number field** and if $s \in \mathbb{C}$ is arbitrary then by $Re(s)$ and $Im(s)$ we mark the **real** and **imaginary** part of s.

Obviously, the transforms $\mathcal{F}, \theta$ and M are **continuous operators** defined on the Banach space B_2 with values in suitable Frechet spaces, according to the inequalities:

$$\sum_{n=1}^{\infty} \mid f(nx) \mid \leq \frac{|| f ||_2 \, \zeta(2)}{x^2},$$

and

$$\mid M(f)(s) \mid \leq || f ||_0 \int_0^1 \mid x^{s-1} \mid dx + || x^{s-2} f(x) ||_0 \int_1^{\infty} \frac{dx}{x^2}.$$

In the sequel we work with the fundamental **Poisson cylinder**

$$\mathbb{P} := \{c \in C^+ : \ c(0) = 1 \ and \ \exists (f \in S_2) with (\hat{f} \in S_2)(c \ = \ f + \hat{f})\} \subset S_2 + S_2 = S_2. \tag{1.4}$$

In particular, for each $c \in \mathbb{P}, \mathcal{F}(c) = c$ i.e. $\mathbb{P}$ is a subcylinder of the eigenvectors space of $\mathcal{F}$, which corresponds to the eigenvalue $+1$, i.e. $\mathbb{P}$ is the set of **fixed points of** $\mathcal{F}$.

It is well-known that the $\mathbb{R}$-vector space C^+ has got the natural structure of the **Frechet space**.

Let $\mathcal{P}$ be the σ-**field** of all subsets of $\mathbb{P}$ generated by **cylinders** C of $\mathbb{P}$ of the form:

$$C \ = \ C(t_1, ..., t_n; B) := \{c \in \mathbb{P} : (c(t_1), ..., c(t_n)) \in B\},$$

where B is a **Borel subset** from $\mathbb{R}^n$ and $t_0 = 0 < t_1 < ... < t_n$.

The importance of the **phase space** $(\mathbb{P}, \mathcal{P})$ is motivated by the two facts :

(1) the functions from the Poisson cylinder $\mathbb{P}$ satisfies the fundamental in the zeta theories **Poisson Summation Formula**(PSF in short), and

(2) On $\mathcal{P}$, there exists fundamental for this short Wiener measure proof - the **non-trivial Wiener-Riemann measure** $r : \mathcal{P} \longrightarrow [0, 1] =: I$.

Our method of transferring results concerning the functional analysis and probability theory of RH looks, broadly speaking, as follows : assume that we have the classical Riemann continuation equation (1.34) and the Riemann hypothesis.

Now having the analytic number theory problem of RH, we can try to solve the corresponding functional analysis and probabilistic problem for the extension of the above mentioned functional equation, for the functional space $\mathbb{P}$, which may be easier than in the original RH setting, since we have additional functional analysis and probability means at our disposal. Having done this, we can again try to put those functional-probability solution together in some way and this may happen to yield a solution of the RH-problem.

A **stochastic process** $B = (B_t : t \geq 0)$ defined on a probability space $(\Omega, \mathcal{A}, Prob)$ is said to be the **standard Brownian motion** iff it is **gaussian** (i.e. its all 1-dimensional **distributions are gaussian**), its **moment function** is zero : $EB_t = 0$, and its correlation function EB_tB_s is equal to $min(t, s)$.

Here and all in the sequel EX marks the **expected value** of a **real random variable** X (rv for short).

For the existence of the Brownian motion see e.g. [W, II.3]. In particular , B has **continuos paths** (since it satisfies the well-known **Kolmogorov condition** - see [W, II.4 , Th.4.5]) and the **independent increments**. Finally B gives the main example of a **Markov process** and a **martingale**.

Let us now consider the **peak function** $p(t)$ defined like :

$$p(t) \ = \ 1 \ - \ t \ \ if \ \ 0 \leq t \leq 1, \tag{1.5}$$

and

$$p(t) \ = \ 0 \, if \ \ t \geq 1,$$

(in particular, p is in Cameron-Martin space) and the stochasic process

$$B^p_t(\omega) := \ (B_t(\omega) \ + \ p(t)) \ \ ; t \geq 0, \omega \in \Omega, \tag{1.6}$$

as a **random element** (re in short) $B^p : (\Omega, \mathcal{A}, Prob) \longrightarrow (C^+, \mathcal{C}^+)$, where $\mathcal{C}^+$ is the **cylinder** σ**- field** of C^+.

Since B determines the **standard Wiener measure** w on the phase space $(C^+, \mathcal{C}^+)$ by the well-known formula (it is the **law** of B) :

$$w(C) \ := \ Prob(B^{-1}(C)), \ \ C \in \mathcal{C}^+, \tag{1.7}$$

then on the Poisson phase space $(\mathbb{P}, \mathcal{P})$ we can define the following **Wiener-Riemann measure** r according to the formula :

$$r(P) \ \ := \ \ \sum_{n=1}^{\infty} \frac{1}{2^n} Prob(\omega \in \Omega : G(t) B^p_{\sqrt{t}}(\omega)(t) \in P, t \in [n-1, n]) \ = \tag{1.8}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} w((G^{-1}P - p) \cap e_n(C[\sqrt{n-1}, \sqrt{n}])), \ P \in \mathcal{P},$$

where

(1) $G(t) = e^{-\pi t^2}$ is the **standard Gauss function**,

(2) $C[\sqrt{n-1}, \sqrt{n}]$ is the Banach space of all real valued continuous functions defined on the segment $[\sqrt{n-1}, \sqrt{n}]$ and considered as the subspace of C^+ through the **embedding** e_n : $C[\sqrt{n-1}, \sqrt{n}] \longrightarrow C^+$ by the formula : $e_n(c)(t) := c(t)$ if $\sqrt{n-1} \leq t \leq \sqrt{n}$; $e_n(c)(t) := c(\sqrt{n-1})$ if $t \leq \sqrt{n-1}$ and $e_n(c)(t) := c(n)$ if $t \geq \sqrt{n}$.

Remark 1 *Let us observe that* e_n *is a linear monomorphism , i.e.* $Ker(e_n) = \{0\}$. *Thus although* $C[\sqrt{n-1}, \sqrt{n}]$ *is not* **formally** *a subspace of* C^+ *(since the product* $C[\sqrt{n-1}, \sqrt{n}] \cap C^+$ *is empty), however we can identify here* $C[\sqrt{n-1}, \sqrt{n}]$ *with its isomorphic image* $Im(e_n) = e_n(C[\sqrt{n-1}, \sqrt{n}])$.

Let us also observe, that exactly therefore since the space (cone) $\mathbb{P}$ *of functions which are self-similar with respect to Fourier transform is a priori "small" for Wiener measure then a posteriori the Wiener-Riemann measure* r *is the* **series** *of measures* $\{r_n\}$ *being the restriction to* $\{C[\sqrt{n-1}, \sqrt{n}]\}$ *of the measure* $(G^{-1})^* w_p$, *which is subsequently the transport by* G^{-1} *of the shifted Wiener measure by* p : $w_p(X) := w(X+p)$ *(which is* **equivalent** *to* w, *since* p *belong to the Cameron- Martin space).*

Proposition 1 *(***On the existence of the Wiener-Riemann measure and its RH-properties***.)*

(I). For each $c \in \mathbb{P}$ *and* $x > 0$ *the* **Poisson Summation Formula***(PSF in short) holds, i.e.*

$$\frac{1}{x}\theta(c)(\frac{1}{x}) \ + \ (c(0) = 1/2x) \ = \ 1/2 \ + \ \theta(c)(x). \tag{1.9}$$

(II). The measure space $(\mathbb{P}, \mathcal{P}, r)$ *has the following four properties :*

*(r_0)(***Non-triviality***).* $r(\mathbb{P}) = 1$.

*(r_1)(***Starting point***).* $\int_{\mathbb{P}} c(0) dr(c) = p(0) = 1$.

*(r_2)(***Vanishing of moments***). For all* $t \geq 1$ *holds*

$$\int_{\mathbb{P}} c(t) dr(c) \ = \ 0.$$

*(r_3)(***The Fubini obstacle-Hardy-Littlewood theorem obstacle - Existence of moments of 1/2-stable Levy distributions***).*

The double integrals (the averaging of the Mellin transform w.r.t. r*)*

$$b_s \ := \ \int\int_{\mathbb{P}\times\mathbb{R}_+} \mid x^{s-1} c(x) \mid dr(c) dx = \int_{\mathbb{P}} M(\mid c \mid)(re(s)) dr(c) \ = \tag{1.10}$$

$$= \ \sum_{n=1}^{\infty} (1/2^n) E[M(\chi_{[\sqrt{n-1}, \sqrt{n}]} G^{-1} B^p_{\sqrt{\cdot}})](re(s)),$$

are

(i) **finite** *if* $Re(s) \in (0, 1/2)$, *and*

*(ii)***infinite** *if* $Re(s) \geq 1/2$.

Proof. (I). It is widely known fact among specialists on zetas. We remark at once that the form (1.9) of (PSF) is a consequence of the fact that each $c \in \mathbb{P}$ is a **fixed point** of $\mathcal{F}$, i.e. $\mathcal{F}(c) = c$. Let $c \in \mathbb{P}$ be arbitrary. Then c is a continuous function in $L_1(\mathbb{R})$. We show that c satisfies the following two conditions :

(i) The series $\sum_{n \in \mathbb{N}} c(x+n)$ is **uniformly convergent** for all $x \in (0,1) =: D$.

Reely, it is nothing that $\theta(c_{+x})(1)$ and obviously

$$max_{x \in D} \mid \sum_{n \geq N} c(x+n) \mid \leq max_{x \in \mathbb{R}} \mid (x+n)^2 c(x+n) \mid \sum_{n \geq N} max_{x \in D}(\frac{1}{(x+n)^2}) \leq \mid\mid c \mid\mid_2 \zeta(2),$$

so, according to the Weierstrass critirion, the above series are uniformly convergent in D. In particular

(ii) $\theta(c)(1) = \sum_{n\in\mathbb{N}^*} c(n)$ is **convergent**.

One of the wider formulations of (PSF) and its proof, in the general case of any LCA groups, a reader can find in the beautiful **Narkiewicz's book** [N, Appendix I. 5, Th. VIII]. We used above Th. VIII in the case : $G = \mathbb{R}, H = K = \mathbb{Z}$ and $D = (0, 1)$.

Now let $x > 0$ be arbitrary. We apply the above (PSF) in the case of the function

$$c_{\cdot x}(y) \ := \ c(xy) \ , \ c \in \mathbb{P}.$$

Since obviously, we have the following easy calculus :

$$\hat{c}_{\cdot x}(y) = 2\int_0^\infty cos(2\pi yz)c(xy)dz = \{u := xz\} = (2/x)\int_0^\infty cos(2\pi\frac{yu}{x})c(u)du =$$

$$= \frac{1}{x}\hat{c}y/x \ = \ \frac{1}{x}c(\frac{y}{x}).$$

Hence

$$\theta(c)(x) \ + \ \frac{1}{2} \ = \ \frac{1}{2x} \ + \ \frac{1}{x}\theta(c)(\frac{1}{x}).$$

$(II)(r_0)$. Let us observe that for each $n \in \mathbb{N}^*$:

$$\mathbb{P} \cap C[\sqrt{n-1}, \sqrt{n}] \ = \ C[\sqrt{n-1}, \sqrt{n}].$$

Really, let $f \in C[\sqrt{n-1}, \sqrt{n}]$ be arbitrary. We can consider f like a restriction of a function $\tilde{f}$ from C^+ with a **compact support**. Let us consider the **second order Fredholm integral equation** of the form (Fox equation) :

$$f(x) \ = \ g(x) \ + \ 2\int_0^\infty cos(2\pi xy)g(y)dy. \tag{1.11}$$

In [KKM, II. 23, Example] the following Fredholm-Fourier-Fox was considered integral equation, with a **parameter** λ :

$$f(x) \ = \ \phi(x) \ - \ \lambda\sqrt{\frac{2}{\pi}}\int_0^\infty \phi(x)cosxtdt. \tag{1.12}$$

The authors solved that Fredholm equation by using the **Mellin transform** and they have obtained the following formula for the solution :

$$\phi(x) \ = \ \frac{f(x)}{1-\lambda^2} \ + \ \frac{\lambda}{1-\lambda^2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)cosxtdt. \tag{1.13}$$

Thus taking $\lambda = -\sqrt{\pi/2} \neq \pm 1$ in (1.12) and (1.13) , we obtain the existence of g with the properties :

$$g(x) \ = \ \frac{\pi f(x)}{\pi - 2} \ + \ \frac{\pi\hat{f}(x)}{\pi - 2},$$

and such that for $x \in [\frac{\sqrt{(n-1)}}{2\pi}, \frac{\sqrt{n}}{2\pi}]$ holds

$$f(2\pi x) \ = \ g(2\pi x) \ + \ \hat{g}(2\pi x).$$

Let us observe that we can put $f(0) = 1$ since $supp(w_p) = C_1[0,1]$ (or equivalently $B_0^p = 1$ with probability 1). Moreover $g \in S_2$ since we have the following trivial calculus :

$$max_{x\in\mathbb{R}} \mid x^2F(x) \mid \leq max_{[0,1]} \mid x^2F(x) \mid \ + max_{x\geq 1} \mid x^2F(x) \mid,$$

for any continuous function $F(x)$. Making the substitution : $u = x^2t$ in the integral $x^2 \int_0^{n(F)} F(t)coxtdt$ where $n(F) > 0$ is such that $supp(f) \subset [0, n(F)]$ we obtain the following easy estimation :

$$\mid x^2 \int_0^{n(F)} F(t)cosxtdt \mid \leq max_{u\geq 0} \mid F(\frac{u}{x^2}) \mid,$$

which is finite and do not depends on x , for $x \geq 1$.

Hence

$$r(\mathbb{P}) = \sum_{n=1}^{\infty} \frac{1}{2^n} Prob(G(\cdot)B^p_{\sqrt{(\cdot)}} \in \mathbb{P} \cap e_nC[n-1,n]) = \sum_{n=1}^{\infty} \frac{w(e_nC[n-1,n])}{2^n} = 1.$$

(r_1). We have

$$\int_{\mathbb{P}} c(0)dr(c) = E(B_0 + p(0)) = p(0) = 1.$$

(r_2). If $t \geq 1$ then $t \in [n-1,n]$ for some $n \geq 2$ and

$$\int_{\mathbb{P}} c(t)dr(c) = G(t)EB_{\sqrt{t}}/2^n = 0.$$

(r_3). This is the unique non-trivial property of r. The idea of this proof is taken from the Proposition 3 of [AM].

We first give the **upper estimation** of the iterated integral

$$I_{dxdr}(s) \ := \ \int_0^{\infty} dx(\int_{\mathbb{P}} \mid x^{s-1}c(x) \mid dr(c)),$$

for s with $u = Re(s) \in (0, 1/2)$.

Let us observe that

$$\int_{\mathbb{P}} \mid c(x) \mid dr(c) \ = \ \frac{G(x)E \mid B^p_{\sqrt{x}} \mid}{2^n}, \tag{1.14}$$

if $n-1 \leq x \leq n, n = 1, 2, ...$ Hence

$$\int_0^{\infty} x^{u-1}dx \int_{\mathbb{P}} \mid c(x) \mid dr(c) \leq \int_0^1 x^{u-1}G(x)p(x)dx + \sum_{n=2}^{\infty} \frac{1}{2^n} \int_{n-1}^{n} x^{u-1}G(x)E \mid B_{\sqrt{x}} \mid dx \leq$$
$$\leq \int_0^1 x^{u-1}G(x)p(x)dx + \int_0^{\infty} x^{u-1}G(x)E \mid B_{\sqrt{x}} \mid dx. \tag{1.15}$$

Since in the inequality (1.15) the first unproper or proper integral obviously exists, thus the problem of the **convergence** of $I_{dxdr}(s)$ is reduced to the convergence of the integral

$$b_s := \int_0^{\infty} x^{u-1}G(x)E \mid B_{\sqrt{x}} \mid dx = 1/\sqrt{2\pi} \int_0^{\infty} x^{u-1}G(x)(\int_{\mathbb{R}} \mid y + y_0 \mid e^{-\frac{(y+y_0)^2}{2x}}/\sqrt{x}dy)dx, \tag{1.16}$$

for any $y_0 > 0$, according to the facts that the distribution of $B_{\sqrt{x}}$ is gaussian with **mean zero** and **variance** $\sqrt{x}$ and the **translational invariance** of the **Lebesgue measure** dy.

Now, let us observe (what was first observed in [AM]), that the iterated integral in the right-hand side of (1.16) - according to :

$$e^{-y^2/2x}e^{-2|y|y_0/2x} \;\leq\; e^{-|y|y_0/x} \tag{1.17}$$

and a suitable substitution can be approximated as follows :

$$\leq (\int_0^\infty x^u \frac{e^{-y_0^2/2x}}{\sqrt{2\pi}x^{3/2}}dx)(y_0^{-2}max_{x\geq 0}(x^2G(x))\int_{\mathbb{R}} \mid t \mid e^{-|t|}dt + max_{x\geq 0}(xG(x))\int_{\mathbb{R}} e^{-|t|}dt). \tag{1.18}$$

But now, the function

$$d_{y_0}(x) \;:=\; \frac{y_0 e^{-y_0^2/2x}}{\sqrt{2\pi}x^{3/2}}\;,\; x > 0, \tag{1.19}$$

is exactly the **density** of a rv L_{y_0} with the **1/2-stable-Levy distribution** (with a parameter y_0). It is well-known that the distribution of L_{y_0} is concentrated on $\mathbb{R}_+$ and that it is the **unique p-stable distribution** with $p \in (0,1)$, which has an elementary analytic and simple formula for the density of a power -exponential form (see e.g. [AM]).

The most important fact concerning L_{y_0}, what we use for this short proof of (RH) is the problem of the **existence of the Orlicz moments** of L_{y_0} (there are also Hilbert moments EX^2 and Banach moments $E \mid X \mid^p, p \geq 1$). More exactly, it is well-known that (see [F]):

$$E(L^u_{y_0}) \;=\; \int_0^\infty x^u d_{y_0}(x)dx < +\infty \; if \; u \in (0,1/2), \tag{1.20}$$

and

$$E(L^u_{y_0}) \;=\; +\infty \; if \; u \geq 1/2. \tag{1.21}$$

Combining (1.18) with (1.20), we finally obtain that the iterated integral $I_{dxdr}(s)$ is **finite** if $Re(s) \in (0,1/2)$. Since obviously the measures r and dx are σ-**finite**, then according to the **Tonelli-Fubini theorem** (TF in short), the numbers b_s are **finite** in this case.

The below lower estimation - also modeled on the previous one - shows that (TF) is **violated** in the case of the triplet :

$$(b_s = \int\int_{\mathbb{P}\times\mathbb{R}} \mid x^{s-1}c(x) \mid dr(c)dx, I_{dxdr}(s), I_{drdx}(s))$$

for $Re(s) \geq 1/2$. Thus the violation of (FT) in the above case is mainly **responsible** for the **non-triviality** of the **Riemann hypothesis** (the **Hardy-Littlewood theorem**: on the critical line $Re(s) = 1/2$ the Riemann zeta $\zeta(s)$ has **infinitely many zeros**).

We have

$$b_{1/2} := \int_0^\infty \frac{dx}{\sqrt{x}}(\int_{\mathbb{P}} \mid c(x) \mid dr(c)) \geq \sum_{n=1}^\infty \frac{1}{2^n}\int_{n-1}^n \frac{G(x)E \mid B^p_{\sqrt{x}} \mid dx}{\sqrt{x}} \geq \tag{1.22}$$

$$\geq \sum_{n=2}^\infty \frac{G(x_n)}{2^n}\int_{n-1}^n \frac{dx}{\sqrt{x}}\int_{\mathbb{R}} \frac{\mid y \mid e^{-y^2/2x}dy}{\sqrt{2\pi x}} =: \sum_{n=2}^\infty g_n I_n \geq$$

But using the classical **Tshebyshev inequality** for positive monotonic finite real sequences (see [Mi, I. 9]) - for each $N \geq 2$ we obtain

$$(\sum_{n=2}^{N+1} g_n I_n) \ \geq (\frac{1}{N}\sum_{n=2}^{N+1} g_n)(\sum_{n=2}^{N+1} I_n), \tag{1.23}$$

where

$$(\sum_{n=2}^{N+1} I_n) \ = \ (2\pi)^{-1/2}\int_{\mathbb{R}} dy \mid y \mid (\int_1^{N+1} \frac{e^{-y^2/2x}dx}{x}). \tag{1.24}$$

Combining (1.22), (1.23) and (1.24) we claim that for all $p \geq 1$ and $N \geq 2$ holds :

$$b_{1/2}^{1/p} \geq (2\pi)^{-1/2p}(\frac{\sum_{n=2}^{N+1} g_n}{N})^{1/p}(\int_{\mathbb{R}} dy \mid y \mid \int_1^{N+1} \frac{e^{-y^2/2x}dx}{x})^{1/p}. \tag{1.25}$$

But it is well-known fact (see e.g. [Mu, II . 5, Exercises 1 and 2]) that from the **Holder inequality** follows that for any $g = (g_2, ..., g_{N+1}) \geq (0, ..., 0)$ the function

$$f(p) \ = \ (\frac{1}{N}\sum_{n=2}^{N+1} g_n)^{1/p}$$

is **non-decreasing** and **bounded** in p and moreover

$$\lim_{p\to\infty}(\frac{1}{N}\sum_{n=2}^{N+1} g_n)^{1/p} = max_{2\leq n\leq N+1}(g_n). \tag{1.26}$$

Combining (1.25) with (1.26) and observing that for each $y \in \mathbb{R}$ the minimum below is not zero:

$$min_{1\leq x\leq N+1} e^{-y^2/2x} \ = \ e^{-y^2/2} > 0, \tag{1.27}$$

we obtain

$$\lim_p b_{1/2}^{1/p} \geq max_{2\leq n\leq N+1}(g_n)(= G(x_2)/4)\lim_p(\int_{\mathbb{R}} \mid y \mid e^{-y^2/2}dy)^{1/p}(\int_1^{N+1}\frac{dx}{x})^{1/p}, \tag{1.28}$$

for each $N \geq 2$. Since the left hand side **does not depend on** N, then we finally get

$$\lim_p b_{1/2}^{1/p} \ \geq \ \frac{G(x_2)}{4}\lim_p(\int_1^{\infty}\frac{dx}{x})^{1/p} = +\infty. \tag{1.29}$$

So, it must be that $b_{1/2} = \infty$ and the Fubini-Tonelli theorem is violated for $Re(s) = 1/2$.

Proposition 2 *(***The Muntz relations for** $(\zeta(s), [s(s-1)]^{-1}, M, \mathcal{F}, \theta)$ *or the* **family of Riemann functional analytic continuation equations for** ζ **and** $\mathbb{P}$*).*

For each $c \in \mathbb{P}$ and s with $Re(s) > 0$ the following functional equation (Rface in short) holds

$$(M(c)\zeta)(s) \ = \ \frac{1}{s(s-1)} \ + \ \int_1^{\infty}(x^{s-1} + x^{-s})\theta(c)(x)dx. \tag{1.30}$$

Proof. Let $c \in \mathbb{P}$ be arbitrary. Since the Mellin transform $M(c)$ is well defined, in this case for $Re(s) \in (0,2)$, then from the definition of the Mellin transform M as the integral, on substituting nx for x under the integral, we have

$$\frac{M(c)(s)}{n^s} = \int_0^\infty c(nx)x^{s-1}dx \ , \ Re(s) \in (0,2). \tag{1.31}$$

Hence, for $Re(s) \in (1,2)$ we obtain that beautiful relation between M, ζ and θ :

$$(M(c)\zeta)(s) = \int_0^\infty \theta(c)(x)x^{s-1}dx = (M \circ \theta)(c)(s). \tag{1.32}$$

Let us observe that the **iterated integral** below

$$\sum_{n=1}^\infty \int_0^\infty x^{u-1} \mid c(nx) \mid dx = \{n^2x = t\} = \zeta(3-u)(max_{t\in[0,1]}sup_n \mid c(t/n) \mid \times$$

$$\times \int_0^1 t^{u-1}dt + max_{t\geq 1} \mid t^2c(t) \mid \int_1^\infty \frac{1}{t^{3-u}}),$$

is absolutely convergent and therefore we can interchange the order of summation and integration. Using the initial condition : $c(0) = 1$, (PSF) and changing variables : $\frac{1}{x} \longrightarrow x$, we can write

$$(M(c)\zeta)(s) = \frac{1}{s-1} - \frac{1}{s} + \int_0^1 x^{s-2}\theta(c)(\frac{1}{x})dx + \int_1^\infty x^{s-1}\theta(c)dx = \tag{1.33}$$

$$\frac{1}{s(s-1)} + \int_1^\infty (x^{-s} \ + \ x^{s-1})\theta(c)(x)dx =: I(\theta(c))(s).$$

The integral on the right-hand side of (33) converges uniformly for $-\infty < a \leq Re(s) < b < +\infty$, since for $x \geq 1$, we have : $\mid x^{-s} \mid \leq x^{-a}$ and $\mid x^{s-1} \mid \leq x^{b-1}$, i.e. because $\hat{c} = c$ and $c \in S_2$ then

$$\theta(c)(x) \leq \sum_{n=1}^\infty \mid c(nx) \mid \leq \frac{max_{x\geq 0} \mid x^2c(x) \mid \zeta(2)}{x^2} \ , \ x \geq 1.$$

Therefore, for each $c \in \mathbb{P}$, the integral $I(\theta(c))(s)$ represents an **entire function** of s. Moreover, since it is well-known that the **classical gamma function** $\Gamma(s) = M(exp^{-1})(s)$ **does not vanish** anywhere, then

$$M(G)(s) = \int_0^\infty x^{s-1}e^{-\pi x^2}dx = \pi^{(1-s)/2}\Gamma(\frac{s+1}{2}) \neq 0.$$

In particular - the belowed $\theta M(G)$-quotient

$$\zeta(s) := \zeta(G,s) := \frac{1}{M(G)(s)s(s-1)} + \frac{I(\theta(G))(s)}{M(G)(s)}, s \in \mathbb{C}, \tag{1.34}$$

gives the **meromorphic continuation** of the local zeta ζ to the whole complex plane.

If now $c \in \mathbb{P} - \{G\}$ (Obviously $G \in \mathbb{P}$), then according to (1.33) we have

$$(M(c)\zeta)(s) = \frac{1}{s(s-1)} + I(\theta(c))(s) \ for \ Re(s) \in (1,2). \tag{1.35}$$

But now, the left-hand side and right-hand of (1.35) (according to the continuation) (1.34)) are the analytic functions in $D := Re(s) \in (0,2) - \{1\}$. Thus, they must be equal in D, according to the **uniqueness** of the analytic continuation of a holomorphic function in a domain.

Theorem 1 *(The Riemann hypothesis)*
If $\zeta(s) = 0$ and $Im(s) \neq 0$ then $Re(s) = 1/2$.

Proof. According to Prop.2

$$Im[(M(c)\zeta)(s)] = \frac{Im(s)(2Re(s)-1)}{\mid s(s-1)\mid^2} + \int_1^\infty [x^{R(s)-1} - x^{-Re(s)}]\theta(c)(x)sin(Im(s)x)dx, \quad (1.36)$$

for $Re(s) \in (0,2)$.

We integrate (1.36) with respect to the **Wiener-Riemann measure** r and obtain :

$$\int_{\mathbb{P}} Im[(M(c)\zeta)(s)]dr(c) = \frac{Im(s)(2Re(s)-1)r(\mathbb{P})}{\mid s(s-1)\mid^2} + Im(\int_{\mathbb{P}} dr(c)\int_1^\infty (x^{s-1}+x^{-s})dx\sum_{n=1}^\infty c(nx). \quad (1.37)$$

According to the property (r_3) of Prop.1 we also have

$$Im(\int_{\mathbb{P}} dr(c)M(c)(s))\zeta(s)) = Im(\sum_{n=1}^\infty (1/2^n)E[GM(\chi_{[\sqrt{n-1},\sqrt{n}]}B^p_{\sqrt{\cdot}})(s)]\zeta(s).$$

For the right-hand side of (1.37) we have an "easy" Fubini-Tonelli theorem: really, let us consider the **triple iterated integrals**:

$$I_{cxr}(\alpha) := \sum_{n=1}^\infty \int_1^\infty x^{\alpha-1}dx\int_{\mathbb{P}} \mid c(nx)\mid dr(c). \quad (1.38)$$

Then, making the substitution : $nx = t$ and using (r_3) of Prop.1 we get

$$I_{cxr}(\alpha) \leq [(\int_1^\infty t^{\alpha-1}G(t)E \mid B_{\sqrt{t}}\mid dt)\zeta(2-\alpha) + (\int_1^\infty t^{-\alpha}G(t)E\mid B_{\sqrt{t}}\mid dt)\zeta(1+\alpha)] = \quad (1.39)$$

$$\zeta(2-\alpha)\int_1^\infty t^{\alpha-1/2}G(t)dt + \zeta(1+\alpha)\int_1^\infty t^{-\alpha+1/2}G(t)dt < +\infty,$$

since $E\mid B_{\sqrt{t}}\mid^2 = t$, the gaussian density G has the moments of arbitrary order and $0 < \alpha < 1/2$.

Thus finally, the averaging of the Muntz's relations from Prop.2 - with respect to the measure r, since by the properties $(r_0 - (r_2))$ of Prop.1 - we have : $\int_{\mathbb{P}} c(0)dr(c) = 1$ and $\int_{\mathbb{P}} c(nx)dr(c) = 0$ for $n \geq 1, x \geq 1$.

Reasuming, we finally obtain

$$Im\{\sum_{n=1}^\infty (1/2^n)E[M(\chi_{[n-1,n]}GB^p_{\sqrt{\cdot}})(s)]\zeta(s)\} = \frac{Im(s)(2Re(s)-1)}{\mid s(s-1)\mid^2}, \ Re(s) \in (0,1/2). \quad (1.40)$$

We calculate very exactly the left-hand of (1.40). For the purposes of that calculus we introduce here the following additional natations :

1. by w^n we denote the standard Wiener measure on the Banach space $C[\sqrt{n-1},\sqrt{n}]$.
2. Let E be any Borel set of $C[\sqrt{n-1},\sqrt{n}]$. Then we denote :

$$w_p^n(E) := w^n(E+p),$$

i.e. w_p^n is the **p-shift** of w^n. Let us remark that $p(x)$ has the derivative $p'(x)$ for a.e. x (with respect to the Lebesgue measure), which is locally-constant function, with support equal to $[0,1]$, so it belongs to $L^2(\mathbb{R}_+)$. In particular, p is from the Cameron-Martin space, and therefore w_p^n is **equivalent to** w^n (we write $w_p^n \sim w^n$), what obviously means that w_p^n is absolutely continuous with respect to w^n and vice versa. Finally, according to the **Girsanov theorem** - the Radon-Nikodem density $\frac{dw_p^n}{dw^n}(c)$ has the form :

$$\frac{dw_p^n}{dw^n}(c) \ = \ e^{-1/2\int_{\sqrt{n-1}}^{\sqrt{n}} p'(x)^2 dx \ - \ \int_{\sqrt{n-1}}^{\sqrt{n}} p'(x) dc(t)}.$$

(Let us mention that the integral with respect to c in the above formula is the Ito integral of deterministic function).

Finally, let us observe that $w_p^n = w^n$ if $n \geq 2$.

3. $(G^{-1})^* w_p^n$ is the **transport** of w_p^n through the **multiplication operator** $m_G : C[\sqrt{n-1}, \sqrt{n}] \longrightarrow C[\sqrt{n-1}, \sqrt{n}]$, where $m_G(c) := G^{-1}c, c \in C[\sqrt{n-1}, \sqrt{n}], G(x) = e^{-\pi x^2}$, i.e.

$$(G^{-1})^* w_p^n(E) \ = \ w_p^n(GE),$$

where E is a Borel set in $C[\sqrt{n-1}, \sqrt{n}]$.

For each $c \in \mathbb{P}$ and $re(s) \in (0,1)$ the Mellin transform $M(c)(s) = \int_0^\infty x^{s-1}c(x)dx$ is well-defined : realy, since $c \in \mathbb{P} \subset S_2$ then

$$| \ M(c)(s) \ |=| \int_0^1 x^{s-1}c(x)dx \ + \ \int_1^\infty x^{s-1}c(x)dx \ |\leq$$

$$\leq max_{x\in[0,1]} \mid c(x) \mid \int_0^1 \frac{dx}{x^{1-re(s)}} \ + \ max_{x\geq 1} \mid x^2 c(x) \mid max_{x\geq 1} x^{re(s)-1} \int_1^\infty \frac{dx}{x^2}.$$

We calculate the iterated integral

$$\int_{\mathbb{P}} dr(c) \int_0^\infty x^{s-1} c(x) dx,$$

where the probability r is the normalized infinite sum of measures $(G^{-1})^* w_p^n$, i.e.

$$r \ = \ \sum_{n=1}^\infty \frac{1}{2^n} (G^{-1})^* w_p^n.$$

According to the bilinearity of the form

$$< r, c >_x := \int_{\mathbb{P}} c(x) dr(c)$$

and the Fubini theorem we have

$$\int_{\mathbb{P}} dr(c) \int_0^\infty x^{s-1}c(x)dx = \int_{\mathbb{P}} d(\sum_{n=1}^\infty 2^{-n}(G^{-1})^* w_p^n)(c) \int_0^\infty x^{s-1}c(x)dx =$$

$$= \int_0^\infty x^{s-1}dx \int_{\mathbb{P}} c(x) d(\sum_{n=1}^\infty 2^{-n}(G^{-1})^* w_p^n)(c) =$$

$$= \int_0^\infty x^{s-1} dx (\sum_{n=1}^\infty 2^{-n} \int_{\mathbb{P}} c(x) d[(G^{-1})^* w_p^n](c)) =$$

$$= \sum_{n=1}^\infty 2^{-n} \int_0^\infty x^{s-1} dx \int_{\mathbb{P} \cap C[\sqrt{n-1},\sqrt{n}] = C[\sqrt{n-1},\sqrt{n}]} \chi_{[\sqrt{n-1},\sqrt{n}]}(x) c(x) d[(G^{-1})^* w_p^n](c),$$

since obviously, from the one-hand side, the support of $[(G^{-1})^* w_p^n]$ is $C[\sqrt{n-1},\sqrt{n}]$ (see also $II(r_0)$) and - from the second-hand side - we can think on $C[\sqrt{n-1},\sqrt{n}]$ that it is embeded isomorphically into the Banach space $B(\mathbb{R}_+)$ of all bounded functions on $\mathbb{R}_+$ through the map : $i_n : C[\sqrt{n-1},\sqrt{n}] \longrightarrow B(\mathbb{R}_+)$ by the formula :

$$i_n(c)(x) \;=\; c(x) \; if \; c \in C[\sqrt{n-1},\sqrt{n}] \; and \; x \in [\sqrt{n-1},\sqrt{n}]$$

and $i_n(c)(x) = 0$ if x is outside of the segment $[\sqrt{n-1},\sqrt{n}]$.

Finally, we thus get

$$\int_{\mathbb{P}} dr(c) \int_0^\infty x^{s-1} c(x) dx = \sum_{n=1}^\infty 2^{-n} \int_{C[\sqrt{n-1},\sqrt{n}]} d[(G^{-1})^* w_p^n](c) M(\chi_{[\sqrt{n-1},\sqrt{n}]} c)(s).$$

But obviously, in the canonical representation - the r.v.$GB^p_{\sqrt{x}}(\omega)$ on $(\Omega, Prob)$ is nothing that r.v. $c(x)$ on $(\mathbb{P}, r)$, i.e. we formally have $GB^p_{\sqrt{x}}(\omega) = c(x)$, where $\omega = c, \omega \in \Omega, c \in \mathbb{P}$.

But, on the other hand we also have (after applying the Fubini theorem)

$$\int_{\mathbb{P}} d[(G^{-1})^* w_p^n](c) \int_0^\infty \chi_{[\sqrt{n-1},\sqrt{n}]}(x) x^{s-1} c(x) dx = \int_0^\infty \chi_{[\sqrt{n-1},\sqrt{n}]}(x) x^{s-1} E(GB^p_{\sqrt{x}}) dx,$$

and - as the consequence - we have

$$\int_{\mathbb{P}} dr(c) \int_0^\infty x^{s-1} c(x) dx = \sum_{n=1}^\infty 2^{-n} M[\chi_{[\sqrt{n-1},\sqrt{n}]} GE(B^p_{\sqrt{\cdot}})](s) =$$

$$= \sum_{n=1}^\infty 2^{-n} \int_0^\infty \chi_{[\sqrt{n-1},\sqrt{n}]}(x) x^{s-1} e^{-\pi x^2} E(B_{\sqrt{x}} + p).$$

Since obviously $E(B_{\sqrt{x}}) = 0$ and $supp(p) = [0,1]$, then we finally get the relation :

$$\int_{\mathbb{P}} dr(c) \int_0^\infty x^{s-1} c(x) dx \;=\; \frac{1}{2} \int_0^1 x^{s-1} p(x) e^{-\pi x^2} dx = \frac{1}{2} M(\chi_{[0,1]} pG)(s).$$

Reasuming, we have showed in fact the following new general **functional equation** for zeta : let $p = p(x)$ be any real valued integrable function with the support in the segment $[0,1]$ and with $p(0) \neq 0$. Then

$$\zeta(s) \;=\; \frac{2p(0)}{s(s-1)M(pG)}(s) \;\; if \;\; re(s) \in (0, 1/2).$$

(Let us remark, that similarly like in the case of the Gamma-zeta-theta relation (1.35) - the quotient in the right-hand side of the above equality does not depend on p).

Additionally, in our case $p(x) = 1 - x, x \in [0,1]$ we have : using the Taylor expansion we obtain that

$$\zeta(s) = \frac{2}{s(s-1)\int_0^1 x^{s-1}(1-x)e^{-\pi x^2}dx}$$

or equivalently, we have the following **refinement Riemann hypothesis**

$$\zeta^{-1}(s) \ = \ \sum_{n=0}^{\infty} \frac{(-\pi)^n s(s-1)}{2n!(s+2n)(s+2n+1)} \ \ if \ \ re(s) \in (0, 1/2).$$

Since the non-trivial Riemann zeta ζ zeros **lay symmetrically with respect to the lines**: (i) critical $Re(s) = 1/2$ and (ii) $Im(s) = 0$ and obviously for $Re(s) > 1$ non-vanishing of $\zeta(s)$ follows from the existence of the **Euler product** whereas for $Re(s) = 1$ from the **de la Vallee-Poussin-Hadamard theorem**, then (1.40) gives the most direct proof and stochastic form of the algebraic geometry conjecture - let us say the **Main Algebraic Hypothesis** (MAH in short) : let us denote:

$$\zeta_t(s) \ := \ Im(s)(2Re(s) - 1)$$

is the **trivial zeta**.

$$\zeta(\mathbb{C}) \ := \ \{s \in \mathbb{C} : \zeta(s) = 0\}$$

is the **zero-dimensional infinite holomorphic manifold** and finally

$$\zeta_t(\mathbb{R}^2) \ := \ \{(x,y) \in \mathbb{R}^2 : \zeta_t(x,y) = 0\}$$

is the **1-dimensional algebraic variete over** $\mathbb{R}$.

The deep sense of the Riemann hypothesis is expressed by the following relation of the **cycles**: $\zeta(\mathbb{C})$ and $\zeta_t(\mathbb{C})$:

$$(MAH) \ \ \ \zeta(\mathbb{C}) \ \subset \zeta_t(\mathbb{C}).$$

2 Final remarks.

(I) Our representation of $\zeta^{-1}(s)$ for $re(s) \in (0, 1/2)$ - which seems to an anonymous referee "much too simple" to be true is analogical to the following - rather simple - series representations of $\zeta(s)$ (see [MJ]) :

$$(i) \ \zeta(s) \ = \ \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \ for \ re(s) > 0 \ and s \neq 1.$$

$$(ii) \ Im(\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)) =: Im(\zeta^*(s)) =$$

$$= Im(s)(1 - 2Re(s)) \cdot (\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-\pi n^2)^j}{j!} \cdot \frac{(4j+1)}{\mid (2j+s)(2j+1-s) \mid^2}),$$

where $re(s) \in (0, 1/2)$, see [ML].

In 1997 **Maślanka**[Ma] proposed a new formula for the zeta Riemann function valid on the whole complex plane $\mathbb{C}$ except the point $s = 1$:

$$\zeta(s) \; = \; \frac{1}{1-s} \sum_{k=0}^{\infty} \frac{A_k \Gamma(k+1-s/2)}{k! \Gamma(1-s/2)}$$

where coefficients A_k are given by

$$A_k \; = \; \sum_{j=0}^{k} (\frac{k}{j})(2j-1)\zeta(2j+2).$$

(II) Instead of the direct approximative proof of the fact that $b_{1/2} = +\infty$ in Prop.1, there is a "purely ideological" proof of that one, based on the **Hardy-Littlewood theorem** , which says that (RH) is **non-trivial** - or equivalently :

$$(HLT) \quad | \, \zeta(\mathbb{C} - \mathbb{R}) \, | = \; +\infty.$$

Really, let us assume that $b_{1/2}$ is **finite**. Then exactly in the same way as in the proof of Th.1, we can deliver the real part Wiener Riemann hypothesis functional equation (rWRfe in short) of the form :

$$Re\{E[M(B^p)(s)]\zeta(s)\} = Re(\frac{1}{s(s-1)}). \tag{2.41}$$

But

$$R(s) := | \, s(s-1) \, |^2 \; Re(s(s-1))^{-1} = Re^2(s) - Re(s) - Im^2(s).$$

Let $R(\mathbb{C})$ marks the **hyperbolic curve** $\{(x,y) \in \mathbb{R}^2 : (x-1/2)^2 - y^2 = (1/2)^2\}$. We thus see that the right-hand side of (2.41) is non-zero if $Re(s) = 1/2$, what means that ζ **has not zeros** on the critical line, i.e. **(RH) would be trivial**, what obviously is not possible, according to the Hardy-Littlewood theorem.

(III) Let us remark that if we simplify the definition of the Wiener-Riemann measure r and define the measure r_∞ on $\mathbb{P}$ by a very similar formula to (WRm)

$$r_\infty(B) := \sum_{n=1}^{\infty} w((B \cap e_n C[n-1, n] - p)), \tag{2.42}$$

then we can easy obtain that the suitable **Wiener numbers**

$$w_s := \int_0^\infty x^{u-1} (\int_{\mathbb{P}} | \, c(x) \, | \, dr_\infty(c)) dx$$

can be easily evaluated like

$$w_s \geq \int_0^\infty dy \, | \, y \, | \, E(L_y^u),$$

and therefore, we immediately get that $w_s = +\infty$ for $Re(s) \geq 1/2$.

Moreover, in the case of r_∞ the following connection between (RH) and the **Orlicz moments** of 1/2-stable-Levy random variables it is better visible - let us say - the following **probabilistic Riemann Hypothesis**(pRH for short) :

(pRH_1) RH, i.e. the statement that $\zeta(s) \neq 0$ if $Re(s) \neq 1/2$ is strictly connected with the **finiteness** of the small moments : $E(L(1/2)^{Re(s)})$ if $Re(s) < 1/2$, for any 1/2-stable Levy rv $L(1/2)$.

(pRH_2) The HLT (i.e. non-triviality of RH), i.e. the statement that for the infinitely many $s = 1/2 + iy$ with $Re(s) = 1/2$ holds : $\zeta(1/2 + iy) = 0$ is strictly associated with the **infinitness** of $E(L(1/2))^{Re(s)}$ if $Re(s) \geq 1/2$, for any 1/2-stable-Levy rv $L(1/2)$.

Finally however, let us observe that r_∞ is **infinite**, i.e. $r_\infty(\mathbb{P}) = \infty$, so there is not any Rhfe connecting $\zeta(s)$ with $\zeta_t(s)$ and any MAH.

(IV) We would try to delete the mentioned in (II) disadvantage of r_∞, considering instead that one - the new probability r_0, subsequently simplyfying (WRm) and defining r_0 simply as the distribution of the Brownian process $(B^p_{\sqrt{t}} : t \geq 0)$, i.e.

$$r_0(A) \ := \ Prob(\omega \in \Omega : B^p_{\sqrt{(\cdot)}}(\omega) \in A).$$

But in this case, according to the **Law of the Iterated Logarithm** (LIL in short) we immediately get that

$$r_0(\mathbb{P}) \ = \ 0,$$

i.e. r_0 is the **trivial measure** and obviously - similarly like r_∞ - it cannot give MAH.

Remark 2 *The positive numbers $\{b_s : s \in \mathbb{C}\}$ play the fundamental role in the Rh-problem. If we denote : $b_s = BM_s(\mathbb{P}, \mathcal{P}, r)$, then we see that b_s is a special case of a much more general construction (see [MB, Sect.4]): let $\mathbb{M}$ be the* **category of all complex valued functional measure spaces with the base** $\mathbb{R}_+$. *Thus - an object of $\mathbb{M}$ is a triple $(M, \mathcal{M}, \mu)$, where $M = F(\mathbb{R}_+, \mathbb{C})$ is a non-zero complex vector space of some functions $f : \mathbb{R}_+ \longrightarrow \mathbb{C}$ endowed with a σ-field $\mathcal{M}$ of subsets of M and $\mu : \mathcal{M} \longrightarrow \mathbb{R}_+$ is a positive σ-additive measure. Let us remark that each "value functional" $v_x : F(\mathbb{R}_+) \longrightarrow \mathbb{C}, v_x(f) := f(x)$ gives a* **quasi-foliation** *of $F(\mathbb{R}_+)$: $F(\mathbb{R}_+) = \cup_{p \in \mathbb{C}} v_x^{-1}(p)$. So, the objects of the category $\mathbb{M}$ can be considered like quasi-foliations. Then, we can define the* **Betti-Mellin numbers** *(see [M4]) by the formula :*

$$BM_s(M, \mathcal{A}, \mu) := \int \int_{M \times \mathbb{R}_+} x^{s-1} f(x) dx d\mu(f).$$

The functors $BM_s(\cdot)$ are **measure invariants**, *i.e. if two measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are* **isomorphic** *in the category $\mathbb{M}$ (we then write $(X, \mathcal{A}, \mu) \simeq (Y, \mathcal{B}, \nu)$), i.e. there exists a set isomorphism $f : X \longrightarrow Y$ with the properties : $\mathcal{A} = f^{-1}(\mathcal{B})$ and $\nu = f^*(\mu)$ (ν is the transport of μ by f), then $BM_s(X, \mathcal{A}, \mu) = BM_s(Y, \mathcal{B}, \nu), s \in \mathbb{C}$.*

Thus, the Betti-Mellin numbers play in the category $\mathbb{M}$ the same important role which the well-known **topological invariants** *: the* **Betti numbers** *$B_i(X)$ and the* **Euler-Poincare characteristic** *$\chi(X)$ of a topological space X in the category of topological spaces $\mathcal{T}$, the Chern numbers in the category of vector bundles or the below mentioned* **foliation invariant** *(see [Fu], [GV] and [Ta]) : let M^n be a linearly connected compact n-dimensional C^s-manifold ($s \geq 4$) with a border or not. Let $\mathcal{F}$ be a C^r-foliation on M^n of codimension 1 and transversal to the border ($r \geq 4$). It is well-known (see [Ta, Th. 7.11]) that on some open covering $\{V_\sigma : \sigma \in \Sigma\}$ of a manifold M^n there exists a differential C^{r-1}-form $\{\omega_\sigma : \sigma \in \Sigma\}$ of order 1, which defines C^{r-1}-grassmanian field $\mathcal{D}(\mathcal{F})$ of tangent (n-1)-dimensional planes of the foliation $\mathcal{F}$. According to the*

famous **Frobenius theorem** *(see [Ta, Sect.28]) on M^n is defined the differential C^{r-2}-form θ of order 1, such that for each form ω_σ on V_σ we have the identity :*

$$d\omega_\sigma \;=\; \omega_\sigma \wedge \theta \;.$$

Subsequently θ defines the differential C^{r-3}-form Γ of degree 3 on M^n by the formula :

$$\Gamma := \theta \wedge d\theta,$$

which is called the **differential Godbillon-Vey form** *of the foliation $\mathcal{F}$. The integral (from a differential form on a manifold):*

$$GV(M^n, \mathcal{F}) \;:=\; \int_{M^n} \Gamma = \int_{M^n} \Gamma(m) dH(m),$$

is a **cobordism foliation invariant** *and is called the* **Godbillon-Vey number** *of a foliated manifold $(M^n, \mathcal{F})$ (here $dH(m)$ is the smooth volume measure on M^n - the Hurwitz measure in the case when M^n is a Lie group).*

Let $\mathcal{FM}^3$ be the category of all oriented 3-dimensional C^s-manifolds with $s \geq 4$ endowed with C^r-foliations of codimension 1 (foliated manifolds). If two foliations $(M_1^3, \mathcal{F}_1)$ and $(M_2^3, \mathcal{F}_2)$ from $\mathcal{FM}$ are **cobordant** *then*

$$GV(M_1^3, \mathcal{F}_1) \;=\; GV(M_2^3, \mathcal{F}_2).$$

The above cobordant invariance of GV-numbers is then used in the deep and famous **Thurston theorem** *: there exist - at least -* **continuum** *cobordism classes in the 3-dimensional and smooth group $\mathcal{F}\Omega_{3,1}^\infty$ of foliation cobordisms of codimension one (let us remark that the group Ω_3 of 3-dimensional and smooth cobordisms is trivial).*

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References

[1] [AM] Albeverio S. and Mądrecki A., *Probability theory and the Riemann hypothesis* , Preprint 2007.

[2] [F] Feller W., *An Introduction to Probability Theory and its Applications II* , John Wiley and Sons Inc., NY 1966.

[3] [Fu] Fuks D. B., *Cohomologies of infinite-dimensional Lie algebras*(in Russian), Moskou, Nauka, 1984.

[4] [GV] Godbillon C. and Vey J., *Un invariant des feuilletages de codimension 1*, C.R. Acad. Sci. 273 (1971), Paris, 92-95.

[5] [KKM] Krasnov M.I., Kisielew A.I. and Makarenko G.I., *Izbranyje glavy vysshej matematiki dla inżenierov i studentov vtuzov*, Nauka, Moskou 1968.

[6] [Ma] Maślanka K., *A hypergeometric-like representation of zeta-function of Riemann* , Cracow Observatory preprint no. 1997/60, (1997), posted at arXiv:math-ph/0105007 2001.

[7] [MB] Mądrecki A., *A short Brownian motion proof of the Riemann hypothesis* , preprint (2004), 1-26.

[8] [ML] Mądrecki A., *The Riemann hypothesis and some stochastic Laplace representation*(preprint in Dissertationes Mathematicae), 1-50.

[9] [MJ] Mądrecki A., *Jacobi probability proof of the Riemann hypothesis* (preprint in Probability and Mathematical Statistics), 1-30.

[10] [M4] Mądrecki A., *4-stable proof of the Riemann hypothesis* (preprint in Scientiae Mathematicae Japonicae), 1-32.

[11] [Mi] Mitrinovic D.S., *Nejednakosti* , Izdavacko Preduzece, Gradeviuska Knjigu, Beograd 1963.

[12] [Mu] Musielak J., *Wstęp do analizy funkcjonalnej* , PWN, Warsaw 1989.

[13] [N] Narkiewicz W., *Elementary and Analytic Theory of Algebraic Numbers* , MM(57), PWN, Warsaw 1974.

[14] [Ta] Tamura I., *Topology of foliations* , Iwanami Shouten, Japan, 1976.

[15] [T] Titchmarsh E., *The theory of the Riemann Zeta-function* , Oxford University Press, London 1951.

[16] [W] Wong E., *Stochastic Processes in Information and Dynamical Systems* , McGraw-Hill, Inc. 1971.

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